

Hamiltonian theory of the half-filled Landau level with disorder: Application to recent NMR data

Ganpathy Murthy*

*Department of Physics and Astronomy,
University of Kentucky, Lexington KY 40506-0055*

R. Shankar†

Department of Physics, Yale University, New Haven, CT 06520

Abstract

The Hamiltonian Theory of the fractional quantum Hall effect is an operator description that subsumes many properties of Composite Fermions, applies to gapped and gapless cases, and has been found to provide results in quantitative accord with data on gaps, relaxation rates and polarizations at temperatures of $300mK$ and above. The only free parameter is λ , which is related to the sample thickness and appears in the Zhang-Das Sarma potential $v(q) = \frac{2\pi e^2}{\kappa q} e^{-ql\lambda}$ where l and κ are the magnetic length and dielectric constant. Here we examine the recent data of Tracy and Eisenstein on the nuclear magnetic resonance relaxation rate at filling factor $\nu = \frac{1}{2}$ deduced from resistivity measurements at temperatures as low as $45mK$. We find that their results can be satisfactorily described by this theory, if in addition to a $v(q)$ with $\lambda \simeq 2$, a constant disorder width $\Gamma \simeq 100mK$ is incorporated.

*Electronic address: murthy@pa.uky.edu

†Electronic address: r.shankar@yale.edu

I. INTRODUCTION

There are several theoretical approaches to the fractional quantum Hall effect (FQHE)[1, 2]. For fractions of the form $\nu = \frac{1}{2p+1}$ one can write down inspired trial wave functions following Laughlin[3] to describe the incompressible liquid ground state and its gapped excitations. These excitations can be further studied using an extension of the Bijl-Feynman approach of Girvin, MacDonald and Platzman [4].

For a more general class of fractions with filling $\nu = p/(2ps+1)$, one can follow Jain[5, 6] and work with Composite Fermions (CF) which are obtained from electrons by attaching $2s$ flux quanta [7, 8]. On average, these attached flux quanta neutralize enough of the external flux so that the total is just right for the CF's to fill exactly p Landau levels. When the wave functions of these unique states are gauge transformed back to electronic language and projected to the Lowest Landau Level (LLL) of electrons, they become excellent trial wave functions. The quasiparticle excitations can likewise be described by starting with particle-hole excitations of these p -filled CF-LL's (Composite Fermion Landau levels).

For the Laughlin fractions Zhang, Hansson, and Kivelson [9] provided a microscopic Chern-Simons (CS) field theory directly linked to the original Hamiltonian. They converted electrons to bosons by adding $2p+1$ flux quanta so that on average the bosons saw no flux. Lopez and Fradkin [10] extended the CS field theory to the Jain fractions $p/(2ps+1)$. In both the bosonic and fermionic cases fluctuations on top of the mean field description were described a CS gauge field. The path integral approach allowed one to more readily compute time and space dependent response functions and go to nonzero temperature T .

The CS theory applied to $\nu = \frac{1}{2}$ leads to CF's that see, on average, zero effective field. Kalmeyer and Zhang[11], and Halperin, Lee, and Read (HLR)[12] studied the Fermi sea of CF's that coupled to CS gauge field. The latter authors performed an exhaustive and definitive study making predictions in agreement with many experiments.

However all the CS theories have one weakness: They do not yield a smooth limit as one projects to the LLL by sending $m \rightarrow 0$. This sends the electronic cyclotron frequency $\omega_c = \frac{eB}{mc} \rightarrow \infty$, thereby forcing all the electrons to be in the LLL (for $\nu \leq 1$). We know that a smooth limit must exist, and that in this limit the kinetic energy should become an ignorable constant and the entire Hamiltonian should be just the Coulomb interaction $v(q)$ between electrons. The quasiparticle mass, its band structure, its residual interactions,

response functions, and so on should be determined solely by $v(q)$. In the Chern-Simons theories sending m to zero causes unavoidable singularities. These theories also lead at the mean-field level to a CF of charge e and not the correct quasiparticle charge of

$$e^* = \frac{e}{2ps + 1} = e(1 - c^2) \quad (1)$$

where

$$c^2 = \frac{2ps}{2ps + 1} \quad (2)$$

is a constant that will appear repeatedly in this paper.

The electric charge does not come out right because the CS procedure attaches the flux but not the screening charge associated with the nucleation of a $2s$ -fold vortex so evident in the wave function description[3, 5]. (In other words, the flux captures only the phase of the vortices but not their modulus, with its $2s$ -fold zero.) In particular, at $\nu = 1/2$, $c^2 = 1$, it does not yield a neutral fermion or describe its dipole structure [13].

The present authors addressed these problems by developing a Hamiltonian approach[14, 15, 16, 17] in which the LLL limit can be taken naturally, and many properties of the CF, such as its reduced charge $e^* = 1/2ps + 1$, and its dipolar nature are encoded unambiguously in the operator structure. Relegating the details of this theory to the next section, we merely point out that it not only reproduces results found by Monte Carlo calculation based on trial wave functions [18], (for potentials that are not too singular at short distances), it also furnishes results in quantitative accord with data on gaps, relaxation rates and polarizations at temperatures of $300mK$ and above for gapped and gapless fractions [19, 20].

The sense in which the Hamiltonian theory accounts for the data needs some elaboration[19]. Consider the $\nu = 1/2$ data of Dementyev *et al*[21], and Khandelwal [22], who measured the polarization and the NMR relaxation rate for relatively high temperatures (down to $300mK$). One can easily find a simple CF theory with an effective mass, and if needed, a Stoner interaction J , to fit the data in a limited range. There are two shortcomings in this approach. First, the origin of the CF mass and interaction are viewed as unrelated when in fact they both originate from the electrostatic interaction between electrons. Next, as emphasized by Dementyev *et al*[21] and Khandelwal[22], if one considers all of their data globally, for both polarization and relaxation, the values of (m, J) needed for one observable are totally incompatible with those mandated by the other. This is because no single pair (m, J) can describe the underlying physics.

On the other hand, in the Hamiltonian theory, the only free parameter is λ , a parameter related to the sample thickness in the Zhang-Das Sarma potential[23]

$$v(q) = \frac{2\pi e^2}{\kappa q} e^{-ql\lambda} \quad (3)$$

where $l = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length, and κ is the dielectric constant. The parameter λ is extracted from one data point, after which the theory does quite well at predicting the rest of the data[19]. As we review the theory, it will become apparent how the CF band structure and interactions (whose nonstandard functional forms are fully determined by $v(q)$) will arise in a correlated way, and vary with the chemical potential, temperature etc., to produce varying effective mass and interaction parameters.

The current work was triggered by recent experiments of Tracy and Eisenstein (TE)[24]. They measured the NMR relaxation rate $\frac{1}{T_1}$ at much lower temperatures than previously (going as low as 45 mK) and for a wide range of B fields while keeping the filling factor at $\nu = \frac{1}{2}$. The relaxation rate was measured by disturbing the system magnetically, and measuring the relaxation of the resistance to its equilibrium value. The precise causal relationship between the non-equilibrium value of polarization and the non-equilibrium value of the resistance is obviously not needed to extract their common relaxation rate.

We do not have much to say about the region of very high B where the polarization has saturated and the spin up and down bands have separated. Here we simply expect the following asymptotic behavior:

$$T_1(B, T) \approx \exp \frac{E_Z - E_Z^*}{T} \quad (4)$$

where E_Z is the Zeeman coupling, and E_Z^* is the critical Zeeman coupling at which the system becomes fully polarized.

Our focus is on the region before saturation where the two spin bands overlap. What does a naive model of noninteracting CF's predict here? One can easily argue that:

$$\frac{1}{T_1 T} \simeq |u(0)|^4 m^{*2} \approx B^{5/3} \quad (5)$$

where m^* is the effective mass (and controls the density of states) and where $u(0)$ measures the height of the wave-function in the transverse direction. Two factors contribute to the final power of B : (i) The density of states per unit volume for each spin species scales as $m^* \simeq B^{1/2}$ (which in turn follows from setting $k_F^2/m^* \simeq e^2/l$) (ii) The factor $|u(0)|^4$, scales as

$B^{2/3}$ because the thickness t of the 2DEG scales as $n^{-1/3} \simeq B^{-1/3}$ [25] and $|u(0)|^2$ integrates to unity in the transverse coordinate. Thus one expects that at fixed T ,

$$T_1 \simeq B^{-5/3}. \quad (6)$$

However the TE data show a nearly flat dependence on B till saturation begins to set in, in contradiction with naive CF theory.

In the Hamiltonian theory the B dependence of T_1 is much more complicated. First the density of states (and hence m^*) varies with the chemical potential since the bands are not quadratic. Next the self-consistent band structure depends on another energy scale, the Zeeman energy, and on the thickness parameter λ , which scales with B as $t/l_0 \simeq B^{1/6}$. All of this precludes any naive scaling. One simply has to turn the crank to see how things depend on B . We did that and here are the main results.

We found that while a suitable $\lambda \simeq 2$ (at 4 Tesla) could produce acceptable graphs, the data clearly called for a nonzero disorder-induced width Γ of order $100mK$ for all momentum states. The existence of such a Γ and its order of magnitude both are implicit in TE data for the equilibrium resistivity[24], reproduced in Fig. (1). Assuming a Drude form for the conductivity

$$\sigma = \frac{ne^2\tau}{m^*} \quad (7)$$

with e being the *electronic charge*, and m^* being the effective mass of the CF's, one finds that the disorder width $\Gamma = \frac{\hbar}{2\tau}$ is a fraction of a Kelvin. Assuming that Γ is roughly independent of n and therefore B , fits the data quite well, as seen in Fig. (1). In past comparisons to data at $300mK$ and higher[19], disorder played a relatively minor role presumably because the temperature was much larger than the Γ of those samples.

In this paper our goal is limited to understanding the effect of disorder on the NMR relaxation rate, and we merely take Eq. (7) as a crude estimate of disorder broadening and do not attempt to provide a complete theory of longitudinal conductance at $\nu = 1/2$, a subject that is yet to be fully developed. (See [12] for a treatment of the charged Chern-Simons fermion coupled to a gauge field.)

Figure 2 shows a comparison of our theory to the T_1 data at 45mK and 100mK. We find that the numerical values $\lambda = 2.00$ and $\Gamma = 100mK$ produce the best overall agreement. A B -independent parallel relaxation channel, with rate $5 \times 10^{-4}s^{-1}$, (representing nuclear spin diffusion out of the 2DEG, and estimated from the TE data), has been added to our theory,

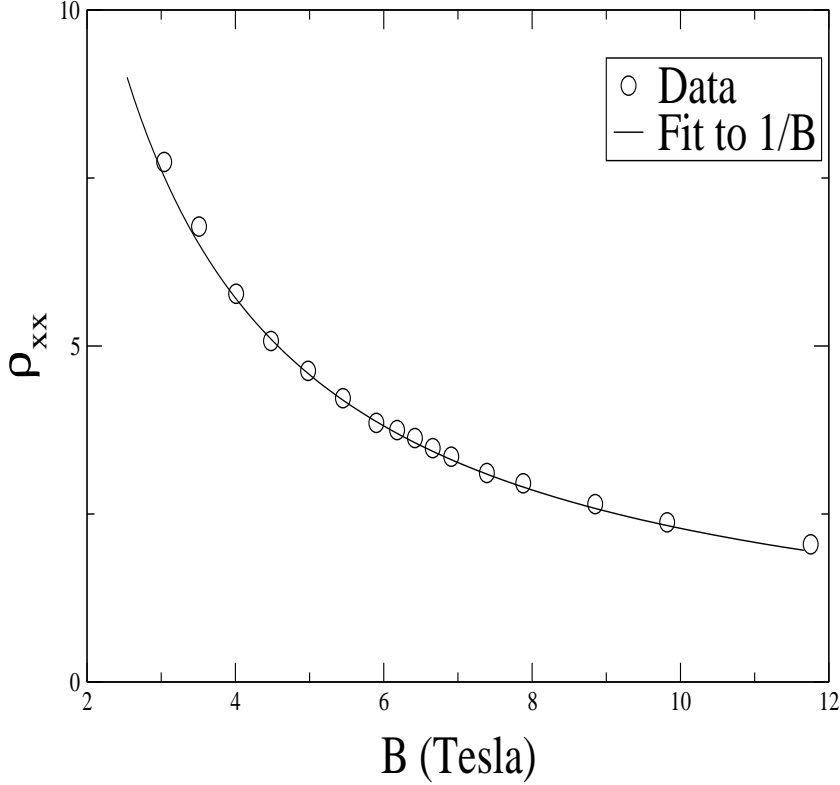


Figure 1: The data from Tracy and Eisenstein, Ref. [24], fitted to a $1/B$ curve. The good fit implies that the transport time is independent of B or n .

and our prediction has been normalized to coincide with the data at $B = 4 \text{ Tesla}$. Finally (B) was scaled from its value at 4 Tesla .

Disorder seems to mimic the effects of nonzero temperature in many ways. We see a clear flattening of the results for T_1 below E_Z^* . Above E_Z^* , since the bottom of the minority spin band is disorder-broadened, there is always some density of states of both spin species near the chemical potential, and therefore T_1 rises much slower than the activated form of Eq. (4).

Figure 3 shows the comparison of theory versus experiment for $1/T_1$ versus T for several B values. As at low T , the theory shows more B -dependence than is measured, and does not do too well past saturation.

The plan of this paper is as follows. In Section II we give a brief review of the Hamiltonian

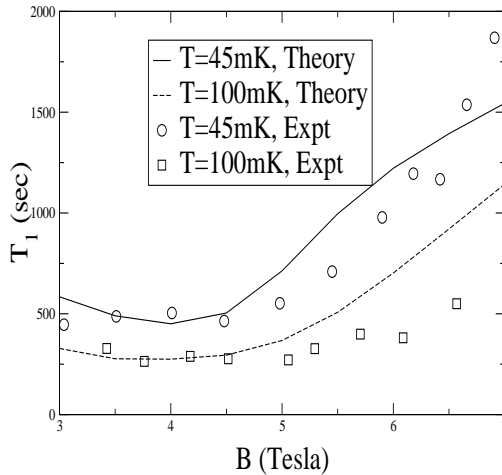


Figure 2: Comparison of theory and experiment for T_1 vs B at $45mK$ (solid line, circles) and $100mK$ (dotted line, squares). The optimal values $\lambda = 2$ (at $4T$) and $\Gamma = 100mK$ were employed. In the comparison, an “offset” of $5 \times 10^{-4}s^{-1}$ has been added to the nuclear relaxation rate to account for the diffusion of nuclear spins out 2DEG into the bulk.

Theory. In Section III we introduce our constant width disorder model, set up the HF equations, and present a formula for the NMR relaxation rate and polarization. Section IV is devoted to a comparison of theory to experiment and the role of our two free parameters, and of the temperature, on the NMR relaxation rate. In Section V we address the effects of Landau-level mixing qualitatively, and in Section VI we conclude by presenting some of the limitations of our work, how they may be overcome, and some open questions.

II. REVIEW OF THE EXTENDED HAMILTONIAN THEORY

We shall furnish only a brief summary of the Hamiltonian theory since complete details may be found in our review article[17]. Our starting point will be the full electronic Hamiltonian

$$H = \sum_i \frac{\Pi_{ei}^2}{2m} + \int \frac{d^2q}{(2\pi)^2} v(q) \rho_e(\mathbf{q}) \rho_e(-\mathbf{q}) \quad (8)$$

where m is the band mass of the electrons, $\Pi_e = \mathbf{p}_e + e\mathbf{A}$ is the velocity operator of the electrons of charge $-e$, $v(q)$ is the Coulomb interaction between the electrons, and

$\rho_e(\mathbf{q}) = \sum_i \exp -i\mathbf{q} \cdot \mathbf{r}_{ei}$ is the electronic density operator.

To facilitate the projection to the LLL, we diagonalize the kinetic energy by decomposing the electron coordinates and momenta into cyclotron ($\boldsymbol{\eta}_e$) and guiding center (\mathbf{R}_e) variables

$$\boldsymbol{\eta}_e = l^2 \hat{\mathbf{z}} \times \boldsymbol{\Pi}_e \quad (9)$$

$$\mathbf{R}_e = \mathbf{r}_e - l^2 \hat{\mathbf{z}} \times \boldsymbol{\Pi}_e \quad (10)$$

where $l = \sqrt{\hbar c / eB}$ is the magnetic length. These coordinates obey the commutation relations

$$[\boldsymbol{\eta}_{ex}, \boldsymbol{\eta}_{ey}] = il^2 \quad [\mathbf{R}_{ex}, \mathbf{R}_{ey}] = -il^2 \quad [\boldsymbol{\eta}_e, \mathbf{R}_e] = 0 \quad (11)$$

The kinetic energy depends only on the cyclotron coordinate, with the LLL corresponding to the harmonic oscillator ground state for this variable. Expressing the electron coordinate as $\mathbf{r}_e = \mathbf{R}_e + \boldsymbol{\eta}_e$ we can now take the LLL limit of the Hamiltonian to get

$$\bar{H}_e = \int \frac{d^2 q}{(2\pi)^2} v(q) \bar{\rho}_e(\mathbf{q}) \bar{\rho}_e(-\mathbf{q}) \quad (12)$$

where $\bar{\rho}_e$ is the electron density operator projected to the LLL.

$$\bar{\rho}_e(\mathbf{q}) = e^{-\frac{1}{4}(ql)^2} \sum_i e^{-i\mathbf{q} \cdot \mathbf{R}_{ei}}. \quad (13)$$

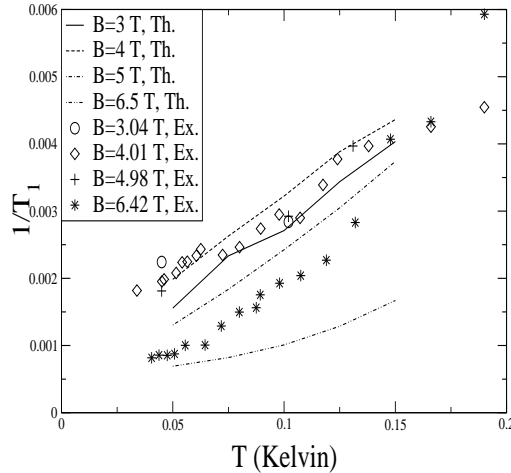


Figure 3: Comparison of theory and experiment for $1/T_1$ vs T . In the comparison, an “offset” of $5 \times 10^{-4} s^{-1}$ has been added to the nuclear relaxation rate to account for the diffusion of nuclear spins out 2DEG into the bulk semiconductor. The parameters used were $\lambda = 2, \Gamma = 100 mK$.

While we can easily project the Hamiltonian to the LLL, working with it is another thing. Here, briefly, are the problems we face and their resolution. The original electronic problem assigns to each particle two coordinates $(x_e, y_e) = \mathbf{r}_e$ and two conjugate momenta $(p_{ex}, p_{ey}) = \mathbf{p}_e$. In the LLL, kinetic energy is quenched but x_e and y_e become conjugate variables so that $H = v(x_e, y_e)$ poses a nontrivial quantum problem.

Equivalently, each electron has a cyclotron coordinate $\boldsymbol{\eta}_e$ and guiding center coordinate \mathbf{R}_e . In the LLL the Hamiltonian is $v(\mathbf{R}_e)$ with R_{ex} and R_{ey} being conjugate variables. Quantizing such a problem is tricky and involves working with analytic wave functions [26].

Our trick was to begin with new particles, the CF's, equal in number to the electrons, each with coordinate \mathbf{r} and velocity $\boldsymbol{\Pi} = \mathbf{p} + e^* \mathbf{A}$, where $e^* = e/(2ps + 1)$ is the CF charge. Thus $\boldsymbol{\Pi}$ describes particles which will fill exactly p LL's. In what follows we will also work with $\boldsymbol{\eta}$ and \mathbf{R} , the corresponding cyclotron and guiding center coordinates of the CF. Note that the CF variables do not carry subscripts.

In this space we form the following entities:

$$\mathbf{R}_e = \mathbf{R} + \boldsymbol{\eta}c = \mathbf{r} - \frac{l^2}{1+c} \hat{\mathbf{z}} \times \boldsymbol{\Pi} \quad (14)$$

$$\mathbf{R}_v = \mathbf{R} + \boldsymbol{\eta}/c = \mathbf{r} + \frac{l^2}{c(1+c)} \hat{\mathbf{z}} \times \boldsymbol{\Pi} \quad (15)$$

The first vector \mathbf{R}_e is identified with the electronic guiding center coordinate since it obeys

$$[R_{ex}, R_{ey}] = -il^2. \quad (16)$$

Thus the LLL projected electronic Hamiltonian we set out to solve is just $H = v(\mathbf{R}_e)$. Since a quantum problem is characterized by the commutation rules, this is a faithful transcription of our original mission. However, there is no assurance the degeneracy of our levels will be the same now.

Indeed we can see there is going to be a huge degeneracy because of the other coordinate \mathbf{R}_v . It describes a particle we call the pseudo-vortex. It commutes with \mathbf{R}_e :

$$[\mathbf{R}_e, \mathbf{R}_v] = 0. \quad (17)$$

and obeys

$$[R_{vx}, R_{vy}] = il^2/c^2, \quad (18)$$

Thus \mathbf{R}_v is a cyclic variable whose dynamics is unrelated to the original problem. Let us get acquainted with it anyway, since all this will change.

First of all, it follows from its commutation rules that \mathbf{R}_v describes a particle of charge $-c^2$ in electronic units. If this object paired with an electron, it would yield an entity with total charge $e(1 - c^2) = e/(2ps + 1) = e^*$. This is exactly how the vortices in the trial wave functions screen the electron to give rise to the CF[5]. However the adjective "pseudo" is appended for two reasons: (i) The vortices in the wave functions are not creatures with their own coordinates independent of electrons. (ii) The pseudo-vortex so far has nothing to do with the original problem, but is rather something we introduced when we enlarged the Hilbert from the LLL projected case described by just \mathbf{R}_e to a regular fermionic space, i.e., the CF space.

Since $H = v(\mathbf{R}_e)$ does not depend on \mathbf{R}_v , the dynamics of \mathbf{R}_v is unspecified. They are like gauge variables and their dynamics is determined by gauge fixing. We made the reasonable choice that the density ρ_v formed out of \mathbf{R}_v annihilated all physical states.

This gives us two options.

The first is to start with a self-consistent Hartree-Fock (HF) solution to the Hamiltonian of Eq. (12) written in CF coordinates, and to compute response functions in a “conserving” approximation[28], such as time-dependent Hartree-Fock (TDHF). In this approximation the constraints are satisfied “weakly”, in correlation functions[29, 30].

The second approximation, which we have employed in the past[14, 15, 19, 20], and use in this manuscript, is more unconventional. We argue that when acting on exact physical states (which are annihilated by $\bar{\rho}_v$) there should be no difference between the projected electron density operator $\bar{\rho}_e$, and the following *preferred density operator*

$$\bar{\rho}_p(\mathbf{q}) = \bar{\rho}_e(\mathbf{q}) - c^2 \bar{\rho}_v(\mathbf{q}). \quad (19)$$

The new operator $\bar{\rho}_p$ has the advantage of exhibiting many of the nonperturbative properties of the CF without any computation, at the tree level. First, it describes a particle of charge $e^* = e(1 - c^2)$ as can be seen by looking at the zeroth order term in an expansion $\rho_p(q)$ in powers of q . The order q term has the correct coefficient to impose another crucial property dictated by Kohn’s theorem[27], that any intra-LLL matrix element of the density operator should vanish faster than linear in q as $q \rightarrow 0$. This property emerges from the conserving calculation only after some effort[29, 30]. It is very striking that one and the same admixture of ρ_v , with coefficient c^2 , serves two purposes at once.

As mentioned this situation is unusual. Normally we have a fixed Hamiltonian and go

in search of a trial HF state from a family, here we have a fixed HF state (with p filled LL of CF's) and go searching among a set of Hamiltonians equivalent in the constrained space. If the constraint is solved exactly, there is nothing to choose between the original $\bar{\rho}_e$, the preferred $\bar{\rho}_p$, or indeed, any arbitrary combination of $\bar{\rho}_e(\mathbf{q})$ and $\bar{\rho}_v(\mathbf{q})$. On the other hand in HF, the constraint is not satisfied and the preferred combination emerges as the best to use since it encodes certain important nonperturbative properties of the CF at leading order.

We will henceforth use the preferred form of the density in all our calculations, to which we now turn.

III. NMR RELAXATION RATE FOR $\nu = \frac{1}{2}$

The assumption that all the nuclei are in thermal equilibrium with each other on a fast time-scale leads to the standard Korringa Law for nuclei in contact with degenerate Fermi gases[31]. The formalism for the clean $\nu = \frac{1}{2}$ system has been previously worked out by one of us[19]. We will proceed directly to the case with disorder. Our model Hamiltonian is the following:

$$H = \int \frac{d^2q}{(2\pi)^2} v(q) \bar{\rho}_p(\mathbf{q}) \bar{\rho}_p(-\mathbf{q}) - E_Z S_z + H_{dis} \quad (20)$$

where E_Z is the Zeeman energy, H_{dis} is the coupling to disorder, and we explicitly present our preferred density operator for $\nu = \frac{1}{2}$

$$\bar{\rho}_p(\mathbf{q}) = e^{-(ql)^2/4} \sum_{\mathbf{k}} -2i \sin\left(\frac{\mathbf{q} \times \mathbf{k} l^2}{2}\right) d_s^\dagger(\mathbf{k}) d_s(\mathbf{k}) \quad (21)$$

Here $d_s^\dagger(\mathbf{k})$ creates a CF in a state with momentum \mathbf{k} and spin-projection s . For our simple model, we do not need to specify the detailed form of the coupling to disorder. We assume that the net effect of H_{dis} in a disorder-averaged treatment is to provide a momentum- and spin-independent width Γ to every single-particle state. We incorporate this into our HF energy calculation as follows. Let the disorder-averaged energy of a state labelled by \mathbf{k}, s be $\epsilon_s(\mathbf{k})$, and its occupation be $n_s(\mathbf{k})$. Then decomposing the Hamiltonian of Eq. (20) in the HF approximation, we find

$$\epsilon_s(\mathbf{k}) = -E_Z \frac{s}{2} + \int \frac{d^2q}{(2\pi)^2} v(q) e^{-(ql)^2/2} [1 - \cos(\mathbf{q} \times \mathbf{k} l^2)] [1 - n_s(\mathbf{k} + \mathbf{q}) - n_s(\mathbf{k} - \mathbf{q})] \quad (22)$$

$$n_s(\mathbf{k}) = \int_{-\infty}^{\infty} d\epsilon' \frac{\Gamma}{\pi(\epsilon'^2 + \Gamma^2)} \frac{1}{1 + e^{\beta(\epsilon_s(\mathbf{k}) - \epsilon' - \mu)}} \quad (23)$$

where $\beta = 1/k_B T$ is the inverse temperature. The form of the first equation is the same as in the clean system, while the disorder-broadening Γ makes itself felt in the occupations. It says that since a particle in a state of momentum \mathbf{k} (defined after averaging) can be in a state of energy different from $\varepsilon(\mathbf{k})$ by an amount of order Γ , a band of energy states of width Γ will contribute to $n(\mathbf{k})$ by convolution.

The spectral function can be related to the disorder average of the exact disorder eigenstates in the HF basis. Let the exact disorder eigenstates be labelled by α . We can expand the operators which annihilate a CF in a momentum state in terms of α :

$$d_s(\mathbf{k}) = \sum_{\alpha} \phi_{\alpha}(\mathbf{k}) d_{s\alpha} \quad (24)$$

Now the retarded single-CF Green's function can be defined as

$$G_s^R(\mathbf{k}, \mathbf{k}', t) = -i\Theta(t) \langle \Omega | \{d_s(\mathbf{k}, t), d_s^{\dagger}(\mathbf{k}', 0)\} | \Omega \rangle \quad (25)$$

Fourier transforming and expanding in the exact disorder states we find

$$G_s^R(\mathbf{k}, \mathbf{k}', \omega) = \sum_{\alpha} \frac{\phi_{\alpha}(\mathbf{k}) \phi_{\alpha}^*(\mathbf{k}')}{\omega - E_{\alpha} + i0^+} \quad (26)$$

Now taking the disorder average and comparing to the spectral function, we infer that

$$\overline{\phi_{\alpha}(\mathbf{k}) \phi_{\alpha}^*(\mathbf{k}')} = \delta_{\mathbf{k}\mathbf{k}'} \frac{\Gamma/\pi}{\Gamma^2 + (E_{\alpha} - \varepsilon_s(\mathbf{k}))^2} \quad (27)$$

When Eqs. (23) are iterated to self-consistency they provide the HF state of the CF-Fermi sea in the presence of disorder. Of course, one needs to maintain the half-filling condition at every HF iteration:

$$\int \frac{d^2k}{(2\pi)^2} [n_{\uparrow}(\mathbf{k}) + n_{\downarrow}(\mathbf{k})] = \frac{1}{4\pi l^2} \quad (28)$$

Now one needs to address the spin correlations, which are important for the NMR relaxation rate[19]. The hyperfine interaction between the electrons and the nuclei is given by

$$\frac{8\pi}{3} \gamma_e \gamma_n \hbar^2 |u(0)|^2 \mathbf{I} \cdot \mathbf{S}_e(0) \quad (29)$$

where $\gamma_{e,n}$ are the magnetic moments of the electron and the nucleus, $u(0)$ is the electronic wavefunction at the nucleus, \mathbf{I} is the nuclear spin, and \mathbf{S}_e is the electronic spin density operator at the position of the nucleus. We define this operator as follows:

$$S_e^a(\mathbf{r} = 0) = \frac{1}{L^2} \sum_{\mathbf{k}, \mathbf{q}} e^{-i\mathbf{q} \times \mathbf{k} l^2 / 2 - (ql)^2 / 4} d_s^{\dagger}(\mathbf{k} - \mathbf{q}) \frac{\sigma_{ss'}^a}{2} d_{s'}(\mathbf{k}) \quad (30)$$

Note that this operator does not commute with the projected density operator, but has LLL spin-charge commutation relations[32]. Now a Fermi golden rule calculation produces the following expression for the NMR relaxation rate[19]:

$$\frac{1}{T_1} = \pi \left(\frac{8\pi\gamma_e\gamma_n}{3} \right)^2 |u(0)|^4 \int_{-\infty}^{\infty} dt \langle \Omega | S_e^x(\mathbf{r}=0,0) S_e^x(\mathbf{r}=0,t) + S_e^y(\mathbf{r}=0,0) S_e^y(\mathbf{r}=0,t) | \Omega \rangle \quad (31)$$

In our evaluation of the spin-correlation function, we will ignore vertex corrections, both due to the gauge field (arising from the imposition of the constraints[29, 30]) and from disorder averaging. We have found from our previous work that the vertex corrections due to the gauge field manifest themselves at extremely low energies, where they give rise to a collective overdamped mode, but are unimportant at not too low temperatures. The vertex corrections due to disorder are important primarily at small \mathbf{q} and at energies much smaller than the disorder width Γ . However, our correlator involves a sum over all \mathbf{q} , and the experiments are carried out at a minimum temperature of $45mK$, which is of the same order as Γ . Thus, we expect the effect of disorder vertex corrections to be negligible as well.

The rest of the calculation is straightforward. Calling the constant $\pi \left(\frac{8\pi\gamma_e\gamma_n}{3} \right)^2 = D$ and performing the disorder average, we have

$$\begin{aligned} \frac{1}{T_1} = & D \frac{|u(0)|^4}{L^4} \sum_{\mathbf{k}_1 \mathbf{k}'_1} e^{i\frac{t^2}{2}(\mathbf{k}_1 \times \mathbf{k}'_1 + \mathbf{k}_2 \times \mathbf{k}'_2) - \frac{t^2}{4}((\mathbf{k}_1 - \mathbf{k}'_1)^2 + (\mathbf{k}_2 - \mathbf{k}'_2)^2)} \times \\ & \int_{-\infty}^{\infty} dt [\overline{\langle \Omega | d_{\uparrow}^{\dagger}(\mathbf{k}_1, t) d_{\uparrow}(\mathbf{k}'_2, 0) | \Omega \rangle \langle \Omega | d_{\downarrow}(\mathbf{k}'_1, t) d_{\downarrow}^{\dagger}(\mathbf{k}_2, 0) | \Omega \rangle} \\ & + \overline{\langle \Omega | d_{\downarrow}^{\dagger}(\mathbf{k}_1, t) d_{\downarrow}(\mathbf{k}'_2, 0) | \Omega \rangle \langle \Omega | d_{\uparrow}(\mathbf{k}'_1, t) d_{\uparrow}^{\dagger}(\mathbf{k}_2, 0) | \Omega \rangle}] \end{aligned} \quad (32)$$

Expressing the single-particle Green's functions in terms of the spectral function, and integrating over t , we obtain

$$\frac{1}{T_1} = D \frac{|u(0)|^4}{L^4} \sum_{\mathbf{k}_1 \mathbf{k}_2} \int_{-\infty}^{\infty} dE n_F(E) (1 - n_F(E)) \left(\frac{\Gamma}{\pi} \right)^2 \times \quad (33)$$

$$\left(\frac{1}{(\Gamma^2 + (E - \epsilon_{\uparrow}(\mathbf{k}_1))^2)(\Gamma^2 + (E - \epsilon_{\downarrow}(\mathbf{k}_2))^2)} + \uparrow \rightarrow \downarrow \right) \quad (34)$$

Now we convert the momentum sums into energy integrals by introducing the densities of states per unit volume $\rho_{\uparrow\downarrow}(\epsilon)$ of the \uparrow and \downarrow spins to get the final expression

$$\begin{aligned} \frac{1}{T_1} = & D |u(0)|^4 \int dE dE_{\uparrow} dE_{\downarrow} e^{-\frac{t^2}{2}(k_{\uparrow}^2 + k_{\downarrow}^2)} I_0(k_{\uparrow} k_{\downarrow} t^2 / 2) \rho_{\uparrow}(E_{\uparrow}) \rho_{\downarrow}(E_{\downarrow}) n_F(E) (1 - n_F(E)) \times \\ & \frac{(\Gamma/\pi)^2}{(\Gamma^2 + (E - E_{\uparrow})^2)(\Gamma^2 + (E - E_{\downarrow})^2)} \end{aligned} \quad (35)$$

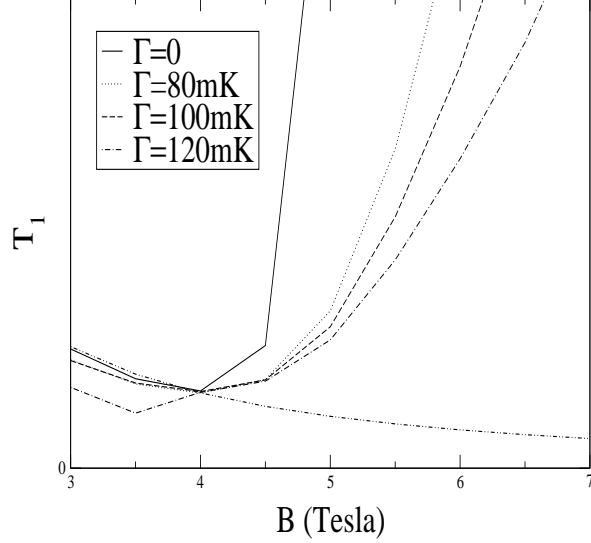


Figure 4: A comparison of the T_1 vs B curves for $\lambda = 2.00$ for different amounts of disorder. The dashed line represents the naive CF prediction of $B^{-5/3}$ below full polarization. T_1 is in arbitrary units and the curves have been normalized to be equal at 4 Tesla for ease of comparison.

Here $k_s = k_s(E_s)$ are to be understood as the result of inverting the energy-momentum dispersion relation, and I_0 is the modified Bessel function arising from the angular average of $\exp(\mathbf{k}_\uparrow \cdot \mathbf{k}_\downarrow l^2/2)$. It is important to note that while the $n_F(E)(1 - n_F(E))$ factor gives the dominant T dependence at low T , there are hidden T dependencies in the energy-momentum dispersion relations obtained from the self-consistent HF solution.

IV. RESULTS AND DISCUSSION

Let us begin by asking how we extract the values for $\bar{1}/T_1$ and Γ . Note that $1/T_1$ has the form

$$\frac{\bar{1}}{T_1} = |u(0)|^4 f(T, (B), \Gamma) \quad (36)$$

Let us look at the T dependence of $\bar{1}/T_1$ at fixed B , say at 4 Tesla. A rough estimate of comes from [19] where the following formula is derived

$$P = .13\sqrt{B(T)^{7/4}} \quad \nu = \frac{1}{2} \quad P < 1 \quad (37)$$

relating the polarization to $B(T)$ in Tesla prior to saturation. From TE data which tells us $P = 1$ occurs first for $B(T) \simeq 6.25$ we deduce $\lambda = 1.9$ as a good start which must be further improved by looking at the data. The width $\Gamma = 100mK$ is found by considering T_1 versus B data for $T = 45mK, T = 100mK$. None of these numbers is unique in that the fit is not perfect and changing the numbers could improve things in one region and worsen it in another. With these values of Γ and λ we pick the overall scale, including $|u(0)|^4$, to agree with the data at 4-Tesla, 45 mK, with the offset of $5 \cdot 10^{-4} s^{-1}$ from spin diffusion into the bulk included. We are now set to make predictions at any other B since Γ is assumed to be constant and the scaling of λ and $|u(0)|^4$ are known:

$$\lambda(B) = \lambda(4Tesla) \left(\frac{B}{4 Tesla} \right)^{1/6} \quad (38)$$

Fig. 4 shows the effect of Γ when λ is held fixed at 2.00 and the temperature is held constant at 45 mK. The dashed line shows the expectations from naive CF theory, which predicts a $B^{-5/3}$ dependence until full polarization, and an exponential increase in T_1 thereafter. As can be seen, even without disorder, the predictions of self-consistent HF vary substantially from the naive CF expectation. However, the qualitative features of the naive expectation are indeed present in the clean limit.

The effect of disorder is to reduce the dependence on B below full polarization, and to make the rise after full polarization gentler than exponential. The second feature can be understood by observing that even when the disorder-averaged position of the bottom of the minority spin band is above the chemical potential, the Lorentzian broadening of the band bottom will produce some density of minority spin states at the chemical potential. Assuming a constant Γ this DOS can be seen to go as $(B - B_c)^{-2}$, which implies a behavior $(B - B_c)^2$ for T_1 .

In Fig. 5 we show the behavior of T_1 in the clean system ($\Gamma = 0$) at three different temperatures. It is clear that at higher temperatures, T_1 is lower due to a greater part of the band being partially occupied. However, the rise of T_1 beyond full polarization is extremely rapid, and it is clear that this is not compatible with the experimental data[24].

A. Polarization

As mentioned earlier, a compelling reason for preferring the Hamiltonian theory over more phenomenological CF theories without a direct link to the electronic Hamiltonian arises when one simultaneously tries to explain data on independent quantities like polarization and $1/T_1$. Where the simpler models call for mutually incompatible values of mass and interaction parameters, the Hamiltonian theory, given just λ from one data point, is able to account for all the salient features of the data.

To this end we are presenting here the predictions of our theory on polarization $P(B, T)$ for the optimal values $\lambda = 2, \Gamma = 100mK$. Since the dependence of P on T is very weak for the range considered by TE, we present just the B dependence in Fig. 6. Unfortunately TE do not measure the polarization. We present these results in the hope that one day they may be tested.

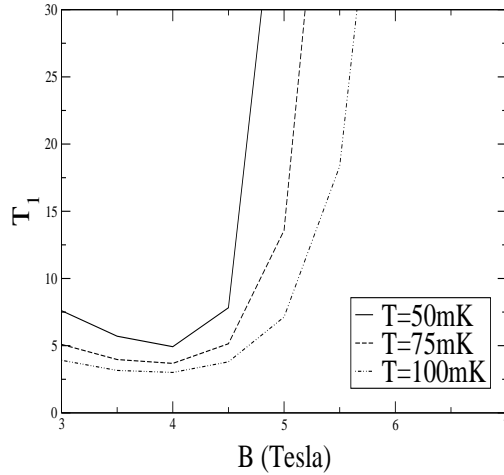


Figure 5: The predictions of the HF approximation of the extended Hamiltonian theory for the clean system for three different temperatures. T_1 increases exponentially beyond full polarization at the lowest temperature of $50mK$. Also note the drop in T_1 below full polarization.

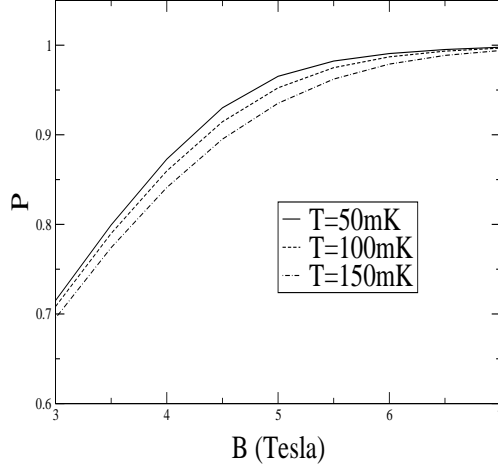


Figure 6: The Polarization $P(B)$ for $\lambda = 2.00$ and $\Gamma = 100mK$.

V. QUALITATIVE EFFECT OF LANDAU-LEVEL MIXING

At fields of the order of a few Tesla the interaction scale $e^2/\kappa l$ (approximately $50\sqrt{B} Kelvin$ where B is in *Tesla*) is comparable to the cyclotron scale $\hbar\omega_c$ (approximately $20B Kelvin$ where B is in *Tesla*). The ratio $\frac{e^2}{\kappa l \hbar\omega_c}$ scales as $1/\sqrt{B}$. The lower the field, the more important Landau-level mixing is[33]. In this section we will show that the qualitative effect of Landau-level (LL) mixing is to improve the agreement between theory and experiment.

In previous work we have developed a formalism for taking Landau-level mixing into account in our Extended Hamiltonian theory[34], where the electronic cyclotron coordinate η_e is retained. Briefly, by a unitary transformation we eliminate the coupling to the higher Landau levels. To leading order, this results in additional four-fermi and six-fermi[34] terms proportional to the perturbation parameter $\zeta = \frac{e^2}{\kappa l \hbar\omega_c}$.

While this can be folded into the HF calculation we have described in previous sections, it is computationally cumbersome. We will proceed to take its effect into account qualitatively. LL-mixing produces level repulsion between the LLL states and higher LL states. The states at the top of the LLL are repelled more, with the net effect being to reduce the interaction-induced bandwidth of the LLL. This means that the density of states of CF's is increased by this effect, implying that T_1 should decrease due to LL-mixing. This reduction is larger at

low fields and smaller at high fields. Applying this insight to Fig. 3 we see that LL-mixing will bring the theoretical prediction into better agreement with the data.

The magnitude of this effect can be estimated by referring to our previous calculations in the gapped fractions[34]. For 3 *Tesla*, the LL-mixing parameter is roughly $\zeta = 1.4$, while for 6 *Tesla* it decreases to 1. From Fig. 1 of Ref. [34] it is seen that the reduction of the FQH gap is between 5-7% in this range. We can thus expect the density of states of the CF's to be enhanced a few percent at 3 *Tesla* relative to the density of states at 6 *Tesla*. This will tend to flatten the concavity of the theoretical curves at low B and improve the agreement with experiment.

VI. CONCLUSIONS, CAVEATS, AND OPEN QUESTIONS

In this paper we have focussed recent experiments of Tracy and Eisenstein[24] on a $\nu = 1/2$ system for temperatures as low as $45mK$ and a range of magnetic fields. They measured the nuclear relaxation rate $1/T_1$ by magnetically disturbing the system and measuring the relaxation of the resistance to its equilibrium value. The predictions of the naive CF picture disagree strongly with the measurements. We showed here that the extended Hamiltonian theory developed by us gives a satisfactory account of the data if in addition to the thickness parameter λ of the electron-electron interaction[23] we incorporate some constant, spin-independent, disorder-induced width of order $\Gamma = 100mK$ for all momentum states.

Our Hartree-Fock calculation ignores vertex corrections due to gauge fields[29, 30] since they affect properties only in the extreme low-energy/low-temperature limit. We also ignore vertex corrections due to static disorder because the property we calculate, $1/T_1$, is an integral over all momenta, and is therefore expected to be insensitive to diffusion-like contributions at small \mathbf{q} . Our optimal value for the disorder width Γ is $100mK - 120mK$, not too different from a crude estimate based on an application of the Drude formula (Eq. (7)) to the measured longitudinal resistivity[24]. This gives us confidence that our calculation is certainly an important part of the explanation.

The value we have used for the thickness parameter in our calculations, $\lambda = 2.00$ at $B = 4$ *Tesla*, requires some discussion. If one took the thickness parameter arising from the self-consistent wavefunction in the transverse direction[23, 25] one would obtain $\lambda \simeq 1$. The primary effect of λ is to soften the Coulomb interaction at short distances. In

incompressible states where there is no linear screening, the finite thickness is indeed the dominant contribution to softening the Coulomb interaction at short distances. However, in a compressible system such as $\nu = \frac{1}{2}$, there will be further screening by the CF's themselves, which will increase the effective value of λ beyond that deduced from the finite thickness of the 2DEG. We believe this is the reason our value of λ is so large.

Let us now turn to some caveats. The fundamental assumption underlying our approach is that disorder averaging gives a true physical picture of the system. While we expect this to be true when the system is far from full polarization and the states of both spins are extended, we expect that more complicated physics is relevant near full polarization. In this regime, we can expect the minority spins to become localized, perhaps in droplets at the minima of the disorder potential. Because they are surrounded by a sea of majority spins, in addition to the usual disorder potential, the minority spins feel an exchange potential which tends to further localize them. Such effects occur in zero-field Fermi liquids as well, but here the scale of the exchange interactions is comparable to the Fermi energy, and exchange effects are magnified. Thus it is doubtful whether disorder-averaging will give a physically relevant answer. We can estimate the behavior of T_1 for large B in this regime as follows: Visualize droplets of minority spin CF's filling the minima of the disorder potential. As the Zeeman energy increases, the droplets will shrink such that their area depends linearly on the Zeeman energy. Since the total relaxation rate should be roughly proportional to the total area of such droplets (relaxation requires that both spin species of CF's be present at a given nuclear site in the localized regime) $1/T_1$ should decrease linearly with the Zeeman energy. Actually, some droplets will disappear when the Zeeman energy increases, so the decrease of $1/T_1$ will be faster than linear in E_Z , with the actual power depending on the distribution of minima of the disorder potential. Note that the disorder potential which should be used here is not the bare one due to inhomogenieties of the dopants, but rather the bare potential screened by the compressible CF state, and including exchange contributions of the type mentioned earlier. A quantitative calculation of these effects is beyond the scope of this paper. In light of these arguments, it is rather fortuitous that our disorder-averaged calculation seems to track the data (Fig. 2) even beyond nominal full polarization.

Finally, we have made no attempt to construct a theory of the longitudinal conductance[12] in the LLL. As has been known for some time[35], the dominant terms in the current operator are Landau-level mixing terms. However, it seems likely that the

low-frequency longitudinal conductance is controlled entirely by LLL physics. An intriguing part of the data of TE[24] is the peak in $\frac{1}{\rho_{xx}} \frac{d\rho_{xx}}{dE_Z}$ near nominal full polarization. Our picture of the formation of localized droplets of minority spin near nominal full polarization may have some relevance for this peak as well. If such droplets are formed, they represent an additional repulsive exchange potential for the *majority* spins, which are mainly responsible for transport. Near nominal full polarization, the system is extremely sensitive to the Zeeman energy, and we expect the scattering due to the exchange potential of these droplets to be maximized. We hope to revisit this issue in future work.

Acknowledgments

We are grateful to Lisa Tracy and Jim Eisenstein for numerous illuminating conversations and for generously sharing their data. R. S. wishes to thank the NSF for partial support under the Grant DMR-0354517.

-
- [1] D. Tsui, H. Stromer and A. Gossard, Phys. Rev. Lett. **48**, 1559, (1982).
 - [2] For a broad survey of the quantum Hall effects, see, “*Perspectives in Quantum Hall Effects*”, S.Das Sarma and A.Pinczuk, Editors (Wiley Interscience, New York 1997).
 - [3] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395, (1983).
 - [4] S. M. Girvin, A. H. MacDonald and P. Platzman, Phys. Rev. **B33**, 2481 (1986).
 - [5] J. K. Jain, Phys. Rev. Lett. **63**, 199, (1989); Phys. Rev. **B 41**, 7653 (1990); Science **266**, 1199 (1994).
 - [6] J. K. Jain and R. K. Kamilla, in Chapter 1 of, “*Composite Fermions*”, Olle Heinonen, Editor (World Scietific, Teaneck, NJ, 1998).
 - [7] J. M. Leinaas and J. Myrheim, Nuovo Cimento **37B**, 1 (1977).
 - [8] S. M. Girvin, in Chapter 9 of, “*The Quantum Hall Effect*”, Edited by R. E. Prange and S. M. Girvin, Springer-Verlag, 1990; S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987).
 - [9] S.-C. Zhang, H. Hansson and S. A. Kivelson, Phys. Rev. Lett. **62**, 82, (1989); D.-H. Lee and S.-C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991); S.-C. Zhang, Int. J. Mod. Phys., **B6**, 25 (1992).

- [10] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991), *ibid* **47**, 7080, (1993), Phys. Rev. Lett. **69**, 2126 (1992).
- [11] V. Kalmeyer and S.-C. Zhang, Phys. Rev. B **46**, 9889 (1992).
- [12] B. I. Halperin, P. A. Lee and N. Read, Phys. Rev. B **47**, 7312 (1993); See also, B. I. Halperin, in *Perspectives in Quantum Hall Effects*, Das Sarma and Pinczuk, Editors (Wiley, New York, 1997).
- [13] N. Read, Semi. Sci. Tech. **9**, 1859 (1994); Surf. Sci., **361/362**, 7 (1996).
- [14] R. Shankar and G. Murthy, Phys. Rev. Lett. **79**, 4437, (1997).
- [15] G. Murthy and R. Shankar, in “*Composite Fermions*”, O. Heinonen, Editor, (World Scientific, Teaneck, NJ, 1998).
- [16] R. Shankar, Phys. Rev. Lett. **83** 2382 (1999).
- [17] G. Murthy and R. Shankar, Rev. Mod. Phys. **75**, 1101 (2003).
- [18] K. Park and J. K. Jain, Phys. Rev. Lett. **80**, 4237 (1998); *ibid* **83**, 5543 (1999).
- [19] R. Shankar, Phys. Rev. **B63**, 085322 (2001).
- [20] G. Murthy, Jour. Phys. Cond. Mat. **12**, 10543 (2000); G. Murthy, Phys. Rev. **B 60**, 13702 (1999).
- [21] A. E. Dementyev, N. Kuzma, P. Khandelwal, S. E. Barrett, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **83**, 5074 (1999).
- [22] P. Khandelwal, Yale Thesis, " Optically pumped NMR in the quantum hall Regimes" 2000, unpublished.
- [23] F.-C. Zhang and S. Das Sarma, Phys. Rev. **B33**, 2903 (1986).
- [24] L. A. Tracy, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, cond-mat/0609618.
- [25] T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).
- [26] S. M. Girvin and T. Jach. Phys. Rev. **B29**, 5617 (1984).
- [27] W. Kohn, Phys. Rev **123**, 1242 (1961).
- [28] G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961); L. P. Kadanoff and G. Baym, “*Quantum Statistical Mechanics*”, Addison-Wesley, Reading, MA, 1989.
- [29] N. Read, Phys. Rev. **B58**, 16262 (1998).
- [30] G. Murthy, Phys. Rev. **B64**, 195310 (2001).
- [31] C. P. Slichter, *Principles of Magnetic Resonance* (Springer, Heidelberg, 1996).
- [32] K. Moon, H. Mori, K. Yang, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka, and S.-C.

- Zhang, Phys. Rev. **B51**, 5138 (1995); K. Yang, K. Moon, L. Belkhir, H. Mori, S. M. Girvin, A. H. MacDonald, L. Zheng, and D. Yoshioka, Phys. Rev. **B54**, 11644 (1996); See also S. M. Girvin and A. H. MacDonald, Chapter 5 of *Perspectives in Quantum Hall Effects*, Das Sarma and Pinczuk, Editors, 1997.
- [33] We are indebted to Jim Eisenstein for encouraging us in a private conversation to consider the effects of LL-mixing on the relaxation rate.
- [34] G. Murthy and R. Shankar, Phys. Rev. **B 65**, 245309 (2002).
- [35] S. L. Sondhi and S. A. Kivelson, Phys. Rev. **B 46**, 13319 (1992).